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On Multivariate Polynomials of Least Deviation from Zero on the Unit Cube

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In the family of all r-variable real polynomials with total degree not exceeding μ and with maximum norm on the unit-cube not exceeding 1, any of the leading coefficients is maximum for a special product of one-variable Chebyshev polynomials of the first kind. This is a consequence of an even more general result on polynomials of least deviation from zero on the unit cube.

1. INTRODUCTION

We deal with polynomials

$$P(x) = \sum c_n x^n, \qquad c_n \in \mathbb{R}, \tag{1.1}$$

in r variables $(x_1, ..., x_r) = x \in \mathbb{R}^r$, $r \in \mathbb{N}$. Here we put

$$x^n = x_1^{n_1} \cdots x_r^{n_r} \qquad \text{for} \quad n = (n_1, ..., n_r) \in \mathbb{N}_0{}^r.$$

Although our particular concern are the real polynomial spaces

$$\mathbb{P}_\mu{}^r := \operatorname{span}\{x^n : |n| \leqslant \mu\},$$

where $\mu \in \mathbb{N}_0$, $|n| = n_1 + \cdots + n_r$ for $n \in \mathbb{N}_0{}^r$, we shall use even the more sophisticated real spaces

$$\Pi_m{}^r := \operatorname{span}\{x^n : |n| \leqslant |m| \text{ or } n_i < m_i \text{ for one } i\}$$

for $m \in \mathbb{N}_0{}^r$.

With T_ν denoting the common Chebyshev polynomial with degree ν of the first kind we can introduce

$$T_m(x) := T_{m_1}(x_1) \cdots T_{m_r}(x_r)$$

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for $m \in \mathbb{N}_0^r$ as its r-variable generalization of degree m. Note that

$$T_m(x) = c_m{}^* x^m + \text{lower degree terms,} \tag{1.2}$$

where

$$c_m{}^* = 2^{|m|-r} \quad \text{for} \quad m > 0 \tag{1.3}$$

(i.e., $m_i > 0$ for all i).

It is well known that, with respect to

$$\| P \| := \max\{| P(x)| : x \in C^r\},$$
$$C^r := [-1, 1]^r, \quad \text{unit cube,}$$

the polynomial T_m is extremal in some sense within the space

$$\mathbb{P}_m{}^r := \text{span}\,\{x^n : n \leqslant m\}.$$

See Ehlich and Zeller [2], Sloss [4] (subspace). We are going to generalize these results for $\mathbb{P}_\mu{}^r$ and even for $\Pi_m{}^r$. Note that $\| T_m \| = 1$ and that

$$T_m \in \mathbb{P}_m{}^r \subset \mathbb{P}_\mu{}^r \subset \Pi_m{}^r \quad \text{for} \quad | m | = \mu. \tag{1.4}$$

Using divided difference methods, we shall prove the following theorems:

THEOREM 1. *Let $m \in \mathbb{N}_0^r$ be fixed. Among all $P \in \Pi_m{}^r$ with the pivot coefficient $c_m = c_m{}^*$, the polynomial $P = T_m$ minimizes the maximum norm on the unit cube C^r.*

THEOREM 2. *Let $m \in \mathbb{N}_0^r$ be fixed. Among all $P \in \Pi_m{}^r$ with maximum norm on the unit cube C^r not exceeding one, the polynomial $P = T_m$ maximizes the pivot coefficient c_m in absolute value.*

We note that, by (1.4), the two theorems remain valid if $\Pi_m{}^r$ is replaced either by $\mathbb{P}_\mu{}^r$, $| m | = \mu$, or by $\mathbb{P}_m{}^r$.

We discuss the problem of uniqueness in Section 3. Here we should point out, however, that Theorem 2 with $\mathbb{P}_\mu{}^r$ instead of $\Pi_m{}^r$ reads more detailed as

COROLLARY 3. *Let $P(x) = \sum_{|m| \leqslant \mu} c_m x^m$, $\| P \| \leqslant 1$. Then $| c_m | \leqslant c_m{}^*$ for $| m | = \mu$.*

We consider it a remarkable fact that, apart from the few cases where $m_i = 0$ for one i, all the leading coefficients c_m, $| m | = \mu$, have the same upper bound $2^{\mu - r}$ in absolute value though this bound is attained for different polynomials, compare (1.3).

2. PROOFS

Let $m \in \mathbb{N}_0{}^r$ be fixed and let $\tilde{\Pi}_m{}^r$ denote the subspace of $\Pi_m{}^r$ where $c_m = 0$. Define

$$\Xi_\nu := \{\xi_0^{(\nu)},..., \xi_\nu^{(\nu)}\}$$

for $\nu \in \mathbb{N}$ to be the set of all critical points of T_ν on the interval $[-1, 1]$ (see Rivlin [3]) and for $\nu = 0$ by $\xi_0^{(0)} := 0$, where these points are assumed to be ordered as follows:

$$\xi_0^{(\nu)} < \xi_1^{(\nu)} < \cdots < \xi_\nu^{(\nu)}.$$

Now define

$$\Xi_m := \Xi_{m_1} \times \Xi_{m_2} \times \cdots \times \Xi_{m_r}$$

for $m \in \mathbb{N}_0{}^r$. Note that Ξ_m is a subset of all critical points of T_m on the unit cube C^r and that, for

$$\tau_n = (\xi_{n_1}^{(m_1)},..., \xi_{n_r}^{(m_r)}) \in \Xi_m \tag{2.1}$$

we have

$$T_m(\tau_n) = (-1)^{|m|+|n|}. \tag{2.2}$$

Now, let $P \in \Pi_m{}^r$ be fixed. We are going to take divided differences from P which act on nodes belonging to Ξ_m only. As can be seen from its effect to monomials, the result of the following process is independent of the order in which the several divided-difference operators are applied. The process is this: For $i = 1, 2,..., r$ we apply the divided-difference operator with respect to the single variable x_i which belongs to Ξ_i as its system of nodes. The result of the whole process is a polynomial $[P]_m$. Note that among all the monomials x^n which span $\Pi_m{}^r$, there is only one for which $[x^n]_m$ is not vanishing. This is the monomial x^m where, by usual Newton–Horner arguments, we obtain

$$[x^m]_m = 1.$$

Hence we have

$$[P]_m = c_m \tag{2.3}$$

for any $P \in \Pi_m{}^r$, $m \in \mathbb{N}_0{}^r$.

On the other hand, it can easily be seen that, as a divided difference, $[P]_m$ has the representation

$$[P]_m = \sum_{\tau \in \Xi_m} \lambda(\tau)\, P(\tau), \tag{2.4}$$

where for the τ_n of (2.1) we have

$$(-1)^{|m|+|n|} \lambda(\tau_n) > 0.$$

This together with (2.2) yields

$$T_m(\tau) \lambda(\tau) > 0 \qquad \text{for all } \tau \in \Xi_m . \tag{2.5}$$

Now we can make the following statements. If we define

$$\textstyle\sum: \Xi_m \to \{-1, +1\}$$

by $\sum(\tau) = \operatorname{sgn} \lambda(\tau)$, then $\sum$ is an extremal signature for $\tilde{\Pi}_m{}^r$ in the sense of Rivlin [3]. This follows from the fact that the right-hand side of (2.4) vanishes for all $P \in \tilde{\Pi}_m{}^r$, which is a consequence of (2.3). On the other hand, we learn by (2.5) that $\sum$ is associated with T_m. Hence, by Rivlin [3, Theorem 2.6], it follows that zero is a best approximation to T_m in $\tilde{\Pi}_m{}^r$ with respect to the maximum norm on the unit cube. This statement is equivalent to the assertion of Theorem 1.

For Theorem 2, assume $P \in \Pi_m{}^r$, $\| P \| \leqslant 1$. If $c_m = 0$, nothing needs to be proved. Now let $c_m \neq 0$. Define

$$\tilde{P} := (c_m{}^*/c_m) P - T_m .$$

Then $\tilde{P} \in \tilde{\Pi}_m{}^r$, hence by Theorem 1

$$| c_m{}^*/c_m | \geqslant | c_m{}^*/c_m | \cdot \| P \| = \| T_m + \tilde{P} \| \geqslant \| T_m \| = 1,$$

and Theorem 2 is proved.

3. Uniqueness

Due to an example of Buck [1], neither in Theorem 1 nor in Theorem 2 can we obtain uniqueness in the general case. This is true even, if we restrict ourselves to $\mathbb{P}_\mu{}^r$, $\mu = | m |$, instead of $\Pi_m{}^r$.

In order to find conditions where uniqueness holds, assume again $P \in \Pi_m{}^r$ to be any polynomials with pivot $c_m = c_m{}^*$. Then

$$\tilde{P} := T_m - P \in \tilde{\Pi}_m{}^r.$$

Now assume

$$\| P \| = \| T_m \|.$$

Then

$$T_m(\tau) \tilde{P}(\tau) \geqslant 0 \qquad \text{for all } \tau \in \Xi_m .$$

Besides, for $\tilde{P}$ we can write (2.4) as

$$0 = \sum_{\tau \in \Xi_m} |\lambda(\tau)|\, T_m(\tau)\, \tilde{P}(\tau)$$

because of (2.5). Together this yields

$$\tilde{P}(\tau) = 0 \qquad \text{for all } \tau \in \Xi_m . \tag{3.1}$$

Now, uniqueness of the solution of the extreme value problem of Theorem 1 would follow, if we could conclude from (3.1) that $\tilde{P}$ is the null polynomial. This is impossible in the general case where $P \in \Pi_m{}^r$ and even in the case where $P \in \mathbb{P}_\mu{}^r$, $\mu = |m|$, as the example of Buck tells us, but possible, if we deal with $\mathbb{P}_m{}^r$ instead of $\Pi_m{}^r$. This result has been found already by Ehlich and Zeller [2].

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